# Magnetic moment effects from time-space non-commutativity in the Dirac equation for Hydrogen atom

Slimane Zaim

Département des Sciences de la Matière, Faculté des Sciences,
Université Hadj Lakhdar – Batna, Algeria.

## Abstract

In this work we study the effect of time-space non-commutativity on the hydrogen atom. Our goal is to solve the Dirac equation for the coulomb potential in a non-commutative time-space up to first order in the non-commutativity parameter using the Seiberg-Witten maps and the Moyal product. We thus find the non-commutative modification of the energy levels of hydrogen atom and show that the non-commutativity is the source of magnetic moment. Comparing the results with experimental values from spectroscopy, we get the bound on non-commutativity parameter at the level of $(0.1\ \mathrm{TeV})^{-2}$.

KEYWORDS: non-commutative field theory, Hydrogen atom, Dirac equation.
PACS NUMBERS: 11.10.Nx, 81Q05, 03.65.Pm

# 1 Introduction

The standard concept of space-time as a geometric manifold is based on the notion of a manifold whose points are locally labelled by a finite number of real coordinates. However, it is generally believed that this picture of space-time as a manifold should break down at very short distances of the order of the Planck length. This implies that the mathematical concepts of high energy physics has to be changed or more precisely our classical geometric concepts may not be well suited for the description of physical phenomenon at short distances [1]. The connection between string theory and the non-commutativity [2, 3, 4, 5] motivated a large amount of work to study and understand many physical phenomenon. The study of this geometry has raised new physical consequences and thus, recently, a non-commutative description of quantum mechanics has stimulated a large amount of research [6, 7, 8, 9, 10, 11] . The non-commutative field theory is characterised by the commutation relations between the position coordinates themselves; namely:

$$[\hat{x}^{\mu}, \hat{x}^{\nu}]_{*} = i\Theta^{\mu\nu}, \tag{1}$$

where $\hat{x}^{\mu}$ are the coordinate operators and $\Theta^{\mu\nu}$ are the non-commutativity parameters of dimension of area that signify the smallest area in space that can be probed in principle. The Groenewald-Moyal star product of two fields $f(x)$ and $g(x)$ is given by:

$$f(x) * g(x) = \exp\left(\frac{i}{2}\Theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}}\right) f(x)\, g(y)\, |_{y=x} \,. \tag{2}$$

The issue of time-space non-commutativity is worth pursuing on its own right because of its deep connection with such fundamental notions as unitarity and causality. Much attention has been devoted in recent times to circumvent these difficulties in formulating theories with $\Theta^{0i} \neq 0$ [12, 13, 14, 15]. There are similar examples of theories with time-space non-commutativity in the literature [16, 17, 18] where unitarity is preserved by a perturbative approach [19].

The most obvious natural phenomena to use in hunting for non-commutative effects are simple quantum mechanics systems, such as the hydrogen atom [20, 21, 22]. In the non-commutative time-space one expects the degeneracy of the initial spectral line to be lifted, thus one may say that non-commutativity plays the role of spin and magnetic field.

In a previous work [23, 24], by solving the deformed Klein-Gordon and Dirac equations in canonical non-commutative space, we showed that the energy is shifted, where the correction is proportional to the magnetic quantum number, which behavior is similar to the Zeeman effect as applied to a system without spin in a magnetic field, thus we explicitly accounted for spin effects in this space.

The purpose of this paper is to study the extension of the Dirac field in canonical non-commutative time-space by applying the result obtained to a hydrogen atom.

This paper is organized as follows. In section 2 we propose an invariant action of the non-commutative Dirac field in the presence of an electromagnetic field. In section 3, using the generalised Euler-Lagrange field equation, we derive the deformed Dirac equation. In section 4, we apply these results to the hydrogen atom, and by the use of the perturbation theory, we solve the deformed Dirac equation and obtain the non-commutative modification of the energy levels. Finally, in section 5, we draw our conclusions.

# 2 Seiberg-Witten maps

Here we look for a mapping $\phi^A \to \hat{\phi}^A$ and $\lambda \to \hat{\lambda}(\lambda, A_\mu)$, where $\phi^A = (A_\mu, \psi)$ is a generic field, $A_\mu$ and $\psi$ are the gauge field and spinor respectively (the Greek and Latin indices denote curved and tangent space-time respectively), and $\lambda$ is the U(1) gauge Lie-valued infinitesimal transformation parameter, such that:

$$\hat{\phi}^A(A) + \hat{\delta}_{\hat{\lambda}} \hat{\phi}^A(A) = \hat{\phi}^A(A + \delta_\lambda A), \tag{3}$$

where $\delta_\lambda$ is the ordinary gauge transformation and $\hat{\delta}_{\hat{\lambda}}$ is a non-commutative gauge transformation which are defined by:

$$\hat{\delta}_{\hat{\lambda}} \hat{\psi} = i\hat{\lambda} * \hat{\psi}, \qquad \delta_\lambda \psi = i\lambda\psi, \tag{4}$$

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_\mu = \partial_\mu \hat{\lambda} + i\left[\hat{\lambda}, \hat{A}_\mu\right]_*, \qquad \delta_\lambda A_\mu = \partial_\mu \lambda. \tag{5}$$

Now in accordance with the general method of gauge theories, in the non-commutative space, using these transformations one can get at second order in the non-commutative parameter $\Theta^{\mu\nu}$ (or equivalently $\Theta$) the following Seiberg–Witten maps [1]:

$$\hat{\psi} = \psi + \Theta\psi^1 + \mathcal{O}\left(\Theta^2\right), \tag{6}$$

$$\hat{\lambda} = \lambda + \Theta\lambda^1(\lambda, A_\mu) + \mathcal{O}\left(\Theta^2\right), \tag{7}$$

$$\hat{A}_\xi = A_\xi + \Theta A^1_\xi(A_\xi) + \mathcal{O}\left(\Theta^2\right), \tag{8}$$

$$\hat{F}_{\mu\xi} = F_{\mu\xi}(A_\xi) + \Theta F^1_{\mu\xi}(A_\xi) + \mathcal{O}\left(\Theta^2\right), \tag{9}$$

where

$$\psi^1(\psi, A) = -\frac{i}{2}\Theta^{\alpha\beta}\left(\{A_\alpha, \partial_\beta\psi\} + \frac{1}{2}\{[\psi, A_\alpha], A_\beta\}\right), \tag{10}$$

$$\lambda^1(\lambda, A) = \Theta^{\alpha\beta}\partial_\alpha\lambda A_\beta, \tag{11}$$

$$A^1_\xi(A) = \frac{1}{2}\Theta^{\alpha\beta}A_\alpha\left(\partial_\xi A_\beta - 2\partial_\beta A_\xi\right), \tag{12}$$

$$F^1_{\mu\xi}(A) = -\Theta^{\alpha\beta}\left(A_\alpha\partial_\beta F_{\mu\xi} + F_{\mu\alpha}F_{\beta\xi}\right), \tag{13}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{14}$$

To begin, we consider an action for a non-commutative Dirac field in the presence of an electrodynamic gauge field in a non-commutative space-time. We can write:

$$\mathcal{S} = \int d^4x \left( \overline{\hat{\psi}} * \left( i\gamma^\nu \hat{D}_\nu - m \right) * \hat{\psi} - \frac{1}{4} \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} \right), \tag{15}$$

where the gauge covariant derivative is defined as: $\hat{D}_\mu = \partial_\mu + ie\hat{A}_\mu$.

Next we use the generic-field infinitesimal transformations (4) and (5) and the star-product tensor relations to prove that the action in eq. (15) is invariant. By varying the scalar density under the gauge transformation and from the generalised field equation and the Noether theorem we obtain [25]:

$$\frac{\partial \mathcal{L}}{\partial \hat{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left( \partial_\mu \hat{\psi} \right)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \left( \partial_\mu \partial_\nu \hat{\psi} \right)} + \mathcal{O}\left( \Theta^2 \right) = 0. \tag{16}$$

# 3 Non-commutative time-space Dirac equation for a Coulomb potential

In this section we study the Dirac equation for a Coulomb interaction $(-e/r)$ in the free non-commutative time-space. This means that we will deal with solutions of the U(1) gauge-free non-commutative field equations [26]. For this we use the modified field equations in eq. (16) and the generic field $\hat{A}_\mu$ in eq. (8), to obtain the free non-commutative field equation:

$$\partial^\mu \hat{F}_{\mu\nu} - ie \left[ \hat{A}^\mu, \hat{F}_{\mu\nu} \right]_* = 0. \tag{17}$$

Using the Seiberg-Witten maps (8)–(9) and the choice (17) , we can obtain the following deformed Coulomb potential [21]:

$$\hat{a}_0 = -\frac{e}{r} - \frac{e^3}{r^4} \Theta^{0j} x_j + \mathcal{O}\left( \Theta^2 \right), \tag{18}$$

$$\hat{a}_i = \frac{e^3}{4\, r^4} \Theta^{ij} x_j + \mathcal{O}\left( \Theta^2 \right). \tag{19}$$

Using the modified field equations in eq. (16) and the generic field $\hat{\psi}$ so that:

$$\delta_{\hat{\lambda}} \hat{\psi} = i\hat{\lambda} * \hat{\psi}, \tag{20}$$

the modified Dirac equation in a non-commutative time-space in the presence of the vector potential $\hat{A}_\mu$ up to the first order of $\Theta$ can be cast into:

$$\left( i\gamma^\mu \partial_\mu - m \right) \hat{\psi} - e\gamma^\mu A_\mu \hat{\psi} - e\gamma^\mu A^1_\mu \hat{\psi} + \frac{ie}{2} \Theta^{ij} \gamma^\mu \partial_i A_\mu \partial_j \hat{\psi} = 0. \tag{21}$$

For a non-commutative time-space $\Theta^{ji} = 0$, where $i, j = 1, 2, 3$ (we do not consider a non-commutative space-space $\Theta^{ji} \neq 0$), in this case it is easy to check

that:

$$i\gamma^{\mu}\partial_{\mu}-m=i\gamma^{0}\partial_{0}+i\gamma^{i}\partial_{i}-m, \tag{22}$$

$$-e\gamma^{\mu}\hat{A}_{\mu}=\frac{e^{2}}{r}\gamma^{0}+\frac{e^{4}}{r^{4}}\gamma^{0}\Theta^{0j}x_{j}, \tag{23}$$

$$\frac{ie}{2}\Theta^{ij}\gamma^{\mu}\partial_{i}A_{\mu}\partial_{j}=-i\frac{e^{2}}{2}\gamma^{0}\frac{\Theta^{0j}x_{j}}{r^{3}}\partial_{0}. \tag{24}$$

Then the non-commutative Dirac equation (21) up to $\mathcal{O}\left(\Theta^{2}\right)$ takes the following form:

$$\left[i\gamma^{0}\partial_{0}+i\gamma^{i}\partial_{i}-m+\frac{e^{2}}{r}\gamma^{0}+\frac{e^{4}}{r^{4}}\gamma^{0}\Theta^{0j}x_{j}-i\frac{e^{2}}{2}\gamma^{0}\frac{\Theta^{0j}x_{j}}{r^{3}}\partial_{0}\right]\hat{\psi}\left(t,r,\theta,\varphi\right)=0. \tag{25}$$

We can write this equation as:

$$\hat{H}\hat{\psi}\left(t,r,\theta,\varphi\right)=i\partial_{0}\hat{\psi}\left(t,r,\theta,\varphi\right). \tag{26}$$

Then replacing:

$$\hat{\psi}\left(t,r,\theta,\varphi\right)=\exp\left(-iEt\right)\hat{\psi}\left(r,\theta,\varphi\right), \tag{27}$$

gives the stationary non-commutative Dirac equation:

$$\hat{H}\hat{\psi}\left(,r,\theta,\varphi\right)=E\hat{\psi}\left(r,\theta,\varphi\right),$$

where $E$ is the ordinary energy of the electron and $\hat{H}$ is the non-commutative Hamiltonian of the form:

$$\hat{H}=H_{0}+H_{\text{pert}}^{\theta}, \tag{28}$$

where $H_0$ is the relativistic hydrogen atom Hamiltonian:

$$H_{0}=\overrightarrow{\alpha}\left(-i\overrightarrow{\nabla}\right)+\beta m-\frac{e^{2}}{r}, \tag{29}$$

and $H_{\text{pert}}^{\theta}$ is the leading-order perturbation:

$$H_{pert}^{\theta}=\left(E-\frac{e^{2}}{r}\right)e^{2}\frac{\overrightarrow{\Theta_{t}}\cdot\overrightarrow{r}}{r^{3}}. \tag{30}$$

The leading long-distance part of $H_{\text{pert}}^{\theta}$ behaves like that of a magnetic dipole potential where the non-commutativity plays the role of magnetic moment. In the above the matrices $\overrightarrow{\alpha}$ and $\beta$ are given by:

$$\beta=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha^{i}=\begin{pmatrix} 0 & \sigma^{i} \\ \sigma^{i} & 0 \end{pmatrix},$$

where $\sigma^i$ are the Pauli matrices:

$$\sigma^{1}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^{2}=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^{3}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To investigate the modification of the energy levels by eq. (30), we use the first-order perturbation theory, where, by restoring the constants $c$ and $\hbar$, the spectrum of $H_0$ and the corresponding wave functions are well known and given by (see [27, 28, 29, 30, 31, 32, 33]):

$$\psi(r,\theta,\varphi) = \begin{pmatrix} \phi(r,\theta,\varphi) \\ \chi(r,\theta,\varphi) \end{pmatrix} = \begin{pmatrix} f(r)\,\Omega_{jlM}(\theta,\varphi) \\ g(r)\,\Omega_{jlM}(\theta,\varphi) \end{pmatrix}, \tag{31}$$

where the bi-spinors $\Omega_{jlM}(\theta,\varphi)$ are defined by:

$$\Omega_{jlM}(\theta,\varphi) = \begin{pmatrix} \mp\sqrt{\frac{(j+1/2)\mp(M-1/2)}{2j+(1\pm1)}}Y_{j\pm1/2,M-1/2}(\theta,\varphi) \\ \sqrt{\frac{(j+1/2)\pm(M+1/2)}{2j+(1\pm1)}}Y_{j\pm1/2,M+1/2}(\theta,\varphi) \end{pmatrix}, \tag{32}$$

with the radial functions $f(r)$ and $g(r)$ given as:

$$\begin{aligned} \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} &= \left(a\frac{mc}{\hbar}\right)^2 \frac{1}{\nu}\sqrt{\frac{\hbar c\left(E\varkappa - mc^2\nu\right)n!}{\left(mc^2\right)^2\alpha\left(\varkappa-\nu\right)\Gamma\left(n+2\nu\right)}}e^{-\frac{1}{2}x}x^{\nu-1}\times \\ &\times \begin{pmatrix} f_1 x L_{n-1}^{2\nu+1}(x) + f_2 L_n^{2\nu-1}(x) \\ g_1 x L_{n-1}^{2\nu+1}(x) + g_2 L_n^{2\nu-1}(x) \end{pmatrix}, \end{aligned} \tag{33}$$

where the ordinary relativistic energy levels are given by:

$$E = E_{n,l} = \frac{mc^2\left(n+\nu\right)}{\sqrt{\alpha^2 + \left(n+\nu\right)^2}}, \qquad n = 0,1,2\cdots \tag{34}$$

and $L_n^{\alpha}(x)$ are the associated Laguerre polynomials [32], with the following notations:

$$\begin{aligned} a &= \frac{1}{mc^2}\sqrt{\left(mc^2\right)^2 - E^2}, \qquad \varkappa = \pm\left(j+\frac{1}{2}\right), \qquad \nu = \sqrt{\varkappa^2-\alpha^2}, \\ f_1 &= \frac{a\alpha}{\frac{E}{mc^2}\varkappa - \nu}, \qquad f_2 = \varkappa - \nu, \qquad g_1 = \frac{a\left(\varkappa-\nu\right)}{\frac{E}{mc^2}\varkappa - m\nu}, \qquad g_2 = \frac{e^2}{\hbar c} = \alpha, \\ x &= \frac{2}{\hbar c}\sqrt{\left(mc^2\right)^2 - E^2}\; r. \end{aligned}$$

In the above $m$ is the mass of the electron and $\alpha$ is the fine structure constant.

### 3.1 Non-commutative corrections to the energy

Now to obtain the modification to the energy levels as a result of the terms (30) due to the non-commutativity of time-space, we use perturbation theory up to the first order. With respect the selection rule $\Delta l = 0$ and choosing the coordinate system $(t,r,\theta,\varphi)$ so that $\Theta^{0j} = -\Theta^{j0} = \Theta\delta^{01}$, we have:

$$\Delta E_{n,l}^{(\Theta)} = \Delta E_{n,l}^{(1)} + \Delta E_{n,l}^{(2)}, \tag{35}$$

where:

$$\Delta E^{(1)}_{n,l} = \frac{E}{\hbar c}e^2 \int_0^{4\pi} d\Omega \int_0^{\infty} drr^{-1}[\psi^{\dagger}_{njlM}\left(r,\theta,\varphi\right)\left(\overrightarrow{\Theta_t}\cdot\overrightarrow{r}\right)\psi_{nj'l'M'}\left(r,\theta,\varphi\right)]$$
$$= \frac{E}{\hbar c}e^2\Theta\langle\frac{1}{r^2}\rangle, \tag{36}$$

and

$$\Delta E^{(2)}_{n,l} = -\frac{e^4}{\hbar c}\int_0^{4\pi} d\Omega \int_0^{\infty} drr^{-2}[\psi^{\dagger}_{njlM}\left(r,\theta,\varphi\right)\left(\overrightarrow{\Theta_t}\cdot\overrightarrow{r}\right)\psi_{nj'l'M'}\left(r,\theta,\varphi\right)]$$
$$= -\frac{e^4}{\hbar c}\Theta\langle\frac{1}{r^3}\rangle, \tag{37}$$

where

$$\langle\frac{1}{r^2}\rangle = 2\hbar c\left(\frac{mc}{\hbar}a\right)^3\left[\frac{\varkappa\left(2E\varkappa - mc^2\right)}{\left(mc^2\right)^2\alpha\nu\left(4\nu^2-1\right)}\right], \tag{38}$$

$$\langle\frac{1}{r^3}\rangle = \left(\frac{mc}{\hbar}a\right)^3\left[\frac{3E\varkappa\left(E\varkappa - mc^2\right) - \left(mc^2\right)^2\left(\nu^2-1\right)}{\left(mc^2\right)^2\nu\left(4\nu^2-1\right)\left(\nu^2-1\right)}\right]. \tag{39}$$

From equation (35) we obtain the modified energy level in non-commutative space-time to the first order of $\Theta$ as:

$$\Delta E^{(\Theta)}_{n,l} = \left(\frac{\alpha}{\hbar c}\right)^2\frac{mc^2a^3}{\nu\left(4\nu^2-1\right)}\times$$
$$\times\left[E\varkappa\left(\frac{2\left(2E\varkappa - mc^2\right)}{\alpha^2} - \frac{3\left(E\varkappa - mc^2\right)}{\left(\nu^2-1\right)}\right) + \left(mc^2\right)^2\right]\Theta. \tag{40}$$

The selection rules for transitions between levels $\left(Nl^M_j \to Nl^{M'}_j\right)$ are $\Delta l = 0$ and $\Delta M = 0, \pm 1$, where $N = n + |\varkappa|$ describes the principal quantum number. The $2P_{1/2}$ and $2P_{3/2}$ levels correspond respectively to:

$$(n = 1, j = 1/2, \varkappa = 1, M = \pm 1/2) \tag{41}$$

and

$$(n = 0, j = 3/2, \varkappa = 2, M = \pm 1/2, \pm 3/2)\,. \tag{42}$$

From eqs. (41), (42) and (40) we can write:

$$\Delta E_{2P_{1/2}} = 7.63637\times 10^{-11}\Theta\ \left(\mathrm{MeV}\right)^3, \tag{43}$$

$$\Delta E_{2P_{3/2}} = 1.27287\times 10^{-10}\Theta\left(\mathrm{MeV}\right)^3. \tag{44}$$

According to ref. [34] the current theoretical accuracy on the $2P$ Lamb shift is about 0.08 kHz. From the splitting (43) and (44), we get the bound:

$$\theta \lesssim \left(0.1\,\mathrm{TeV}\right)^{-2} \qquad \text{or} \qquad \theta \lesssim \left(0.09\,\mathrm{TeV}\right)^{-2}. \tag{45}$$

Restoring the constants $c$ and $\hbar$ in eq. (45) we write the bound on the non-commutativity parameter as:

$$\Theta \lesssim 1.1 \times 10^{-34}\mathrm{m}^2. \tag{46}$$

It is interesting that the value of the upper bound on the time-space non-commutativity parameter as derived here is better than the results of refs. [19, 20, 35] . This value is only in the sense of an upper bound and not the value of the parameter itself, which is close to the minimum scale compatible with the quantum mechanical approach of about $10^{-30}\mathrm{m}^2$.

# 4 Conclusions

In this work we proposed an invariant non-commutative action for a Dirac particle under the generalised infinitesimal gauge transformations. Using the Seiberg-Witten maps and the Moyal product, we derived the modified Dirac equation for a Coulomb potential to the first order of $\Theta$ and we showed that the non-commutativity plays the role of magnetic moment. Using perturbation-theory methods to first order, we derived the non-commutative corrections to the energy levels and the lamb-shift was induced. We derived a new bound on the time-space non-commutative parameter around $(0.1\mathrm{TeV})^{-2}$.


# Acknowledgments

This work is supported by the LEPA and PNR research project.


# References


[1] S. Doplicher, K. Fredenhagen and J.E. Roberts, Phys. Lett. B331 (1994) 39; S. Doplicher, K. Fredenhagen and J.E. Roberts, Commun. Math. Phys. 172 (1995) 187; T. Yoneya, Progr. Theor. Phys.103 (2000) 1081, [hep-th/0004074].

[2] N. Seiberg and E. Witten, JHEP 9909 (1999) 032.

[3] E. Bergshoeff, D. S. Berman, J. P. van der Schaar and P. Sundell, Nucl. Phys. B 590 (2000) 173.

[4] S. Kawamoto and N. Sasakura, JHEP 0007 (2000) 014.

[5] A. Das, J. Maharana and A. Melikyan, JHEP 0104 (2001) 016.

[6] Cai, S., Jing, T., Guo, G., Zhang, R.: J. Theor. Phys. 49, 1699 (2010)

[7] Dulat, S., Li, K.: Chin. Phys. C 32, 32 (2008)

[8] Gamboa, J., Mondez, F., Loewe, M., Rojas, J.C.: Mod. Phys. Lett. A 16, 2075 (2001)

[9] Li, K., Dulat, S.: Eur. Phys. J. C 46, 825 (2006)

[10] Li, K., Chamoun, N.: Chin. Phys. Lett. 24, 1183 (2007)

[11] Dulat, S., Li, K.: Mod. Phys. Lett. A 21, 2971 (2006)

[12] S. Doplicher, K. Fredenhagen, J. Roberts, Phys. Lett. B 331 (1994) 39.

[13] S. Doplicher, K. Fredenhagen, J. Roberts, Comm. Math. Phys. 172 (1995) 187, [hep-th/0303037].

[14] O. F. Dayi, B. Yapiskann, JHEP 10 (2002) 022, [hep-th/0208043].

[15] O. Bertolami and L. Guisado, JHEP 0312 (2003) 013.

[16] C. S. Chu, J. Lukierski, and W. J. Zakrzewski, Nucl. Phys. B632(2002) 219, [hep-th/0201144].

[17] D. A. Eliezer and R. P. Woodard, Nucl. Phys. B 325 (1989) 389.

[18] T. C Cheng, P. M. Ho, and M. C. Yeh, Nucl. Phys. B 625 (2002) 151.

[19] A. Saha, Time-Space Non-commutativity in Gravitational Quantum Well scenario, Eur.Phys.J. C51, (2007)199 .

[20] M. Chaichian, M. M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett. 86 (2001) 2716.

[21] T. C. Adorno, M. C. Baldiotti, M. Chaichian, D. M. Gitman, A. Tureanu, Phys. Lett. B682 (2009) 235.

[22] H. Motavalli, A. R. Akbarieh, Mod. Phys. Lett. A25 (2010) 2523.

[23] S. Zaim, L. Khodja and Y. Delenda, IJMPA 23 (2011) 4133.

[24] L. Khodja, S. Zaim, IJMPA Vol. 27, No. 19 (2012) 1250100

[25] N. Mebarki, S. Zaim, L. Khodja and H. Aissaoui, Phys. Scripta 78 (2008) 045101.

[26] A. Stern, Phy. Rev. Lett. 100, (2008) 061601.

[27] A. Akhiezer and V. B. Berestetskii, Quantum Electrodynamics, Interscience Publishers, New York, 1965.

[28] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, Relativistic Quantum Theory, Pergamon Press, Oxford, 1971.

[29] H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One- and Two-Electron Atoms, Springer–Verlag, Berlin,1957; reprinted by Dover, New York, 2008.

[30] C. Itzykson and J.-B. Zuber, Quantum Field Theory, Dover Publications, New York, 2005.

[31] A. Messiah, Quantum Mechanics, Vol. 2, North-Holland, Amsterdam, 1961; reprinted by Dover, New York,1999.

[32] A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics, Birkhäuser, Basel, Boston, 1988.

[33] S. K. Suslov, J. Phys. B 42 (2009) 185003, ArXiv: 0906.3338 [quant-ph].

[34] M. I. Eides, H. Grotchand, V. A. Shelyuto, Phys. Rep.342 (2001) 63 .

[35] S. M. Carroll, J. A. Harvey, V. Alan Kostelecky, C. D. Lane, T. Okamoto, Phys.Rev.Lett.87:141601,2001